

# DEFORMATION OF HYPERBOLIC MANIFOLDS IN $\mathrm{PGL}(n, \mathbb{C})$ AND DISCRETENESS OF THE PERIPHERAL REPRESENTATIONS

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**ABSTRACT.** Let  $M$  be a cusped hyperbolic 3-manifold, e.g. a knot complement. Thurston [Thu79] showed that the space of deformations of its fundamental group in  $\mathrm{PGL}(2, \mathbb{C})$  (up to conjugation) is of complex dimension the number  $\nu$  of cusps near the hyperbolic representation. It seems natural to ask whether some representations remain discrete after deformation. The answer is generically not. A simple reason for it lies inside the cusps: the degeneracy of the peripheral representation (i.e. representations of fundamental groups of the  $\nu$  peripheral tori). They indeed generically become non discrete, except for a countable set. This last set corresponds to hyperbolic Dehn surgeries on  $M$ , for which the peripheral representation is no more faithful.

We work here in the framework of  $\mathrm{PGL}(n, \mathbb{C})$ . The hyperbolic structure lifts, via the  $n$ -dimensional irreducible representation, to a representation  $\rho_{\mathrm{geom}}$ . We know from the work of Menal-Ferrer and Porti [MFP11] that the space of deformations of  $\rho_{\mathrm{geom}}$  has complex dimension  $(n-1)\nu$ .

We prove here that, unlike the  $\mathrm{PGL}(2)$ -case, the generic behaviour becomes the discreteness (and faithfulness) of the peripheral representation: in a neighbourhood of the geometric representation, the non-discrete peripheral representations are contained in a real analytic subvariety of codimension  $\geq 1$ .

## 1. INTRODUCTION

Let  $M$  be a complete orientable hyperbolic 3-manifold with  $\nu \geq 1$  cusps, e.g. a knot complement. For a Lie group  $G$ , consider the space  $\chi(M, G)$  of representations of its fundamental group modulo conjugacy:

$$\chi(M, G) = \mathrm{Hom}(\pi_1(M), G)/G.$$

Let  $T_1, \dots, T_\nu$  be the peripheral tori of  $M$  (so that the cusps are of the form  $T_i \times [0, \infty)$ ). We choose once for all a longitude  $l_i$  and a meridian  $m_i$  for each of them.

In the case  $G = \mathrm{PGL}(2, \mathbf{C})$ , a natural point to consider in the space  $\chi(M, \mathrm{PGL}(2, \mathbf{C}))$  is the (class of the) monodromy  $[\rho_{\mathrm{hyp}}]$  of the hyperbolic structure. Following Thurston [Thu79], Neumann-Zagier [NZ85] showed it is a smooth point and the complex dimension of  $\chi$  at this point equals to the number  $\nu$  of cusps (see also [Kap01]). Moreover the (hyperbolic) length of the longitudes  $l_i$  – or directly related parameters as the trace of their holonomies – are natural local parameters for this space. In other words, in a neighbourhood of  $[\rho_{\mathrm{hyp}}]$ , the deformations are described by their restriction on the tori. This restriction, called peripheral representation, will be a central object in this paper:

**Definition 1.** *For any  $\rho \in \chi(M, G)$ , its peripheral representation  $\rho_{\mathrm{periph}}$  is the collection of the restrictions of  $\rho$  to  $\pi_1(T_i)$ .*

The above mentioned phenomenon – that the peripheral representations prescribe the whole representation – is called *local rigidity* (around  $[\rho_{\mathrm{hyp}}]$ ).

Still in the  $\mathrm{PGL}(2, \mathbf{C})$ -case, we know which representations remain discrete in a neighbourhood of the hyperbolic one [Thu79, NZ85]. Indeed, after deformation of the hyperbolic representation, the new representation becomes generically non discrete. And a simple reason for it lies in the peripheral representations: they already are not discrete, except for a countable set. This last set corresponds to hyperbolic Dehn surgeries on  $M$  (or a finite covering) for which the whole representation is indeed discrete. Beware that in this situation the peripheral representations are no more faithful.

We work in this paper in the framework of  $\mathrm{PGL}(n, \mathbf{C})$ . The hyperbolic structure lifts, via the  $n$ -dimensional irreducible representation

$$r_n : \mathrm{PGL}(2, \mathbf{C}) \rightarrow \mathrm{PGL}(n, \mathbf{C}),$$

to an irreducible representation  $\rho_{\mathrm{geom}} = r_n \circ \rho_{\mathrm{hyp}}$  called the *geometric representation*. Let us mention that, when  $n = 3$ , the irreducible representation  $r_3$  is more widely known as the adjoint representation  $\mathrm{Ad}$ . The problem of local rigidity around  $\rho_{\mathrm{geom}}$  has already been studied by Menal-Ferrer and Porti in [MFP11] and shown to hold also for  $G = \mathrm{PGL}(n, \mathbf{C})$ , see theorem 1. We hence know that the space of deformations near  $\rho_{\mathrm{geom}}$  has complex dimension  $(n - 1)\nu$  and that the symmetric functions of the eigenvalues of  $\rho(l_i)$  are local parameters for  $\chi(M, \mathrm{PGL}(n, \mathbf{C}))$  – see fact 1 for a precise statement. Let us also mention that the paper [BFG12a] recovers this theorem for  $n = 3$ . Its approach to the problem, following [BFG12b], leads to actual computations in  $\chi(M, \mathrm{PGL}(3, \mathbf{C}))$ . In the last section, we will present the example of the 8-knot complement.

We prove here that the peripheral representations are generally discrete: in a neighbourhood of the geometric representation, the non-discrete peripheral representations are contained in a real analytic subvariety, see theorem 2. The motto for the proof is that,  $\mathrm{PGL}(n, \mathbf{C})$  being of rank  $n - 1$ , there is enough room to construct discrete and faithful representations of the commutative groups  $\pi_1(T_i)$  as soon as  $n \geq 3$ . It is worth insisting here on the fact that this motto should be carefully implemented. The examples show that peripheral discreteness does not hold generically around any unipotent representation. This is why we concentrate our work on the geometric representation even if the techniques may be used to deal with other unipotent representations.

It raises an interesting question: in the  $\mathrm{PGL}(2, \mathbf{C})$ -case, in the neighbourhood of the hyperbolic structure, the peripheral representations are discrete if *and only if* the whole representation corresponds to a hyperbolic Dehn filling (or a ramified covering) and is therefore discrete. So there is a local equivalence between the peripheral discreteness and the discreteness of the whole representation<sup>1</sup>. Does the same hold in  $\mathrm{PGL}(n, \mathbf{C})$  ? It would have the surprising consequence that generically, in a neighbourhood of the geometric representation, the deformed representation remains discrete.

## 2. PERIPHERAL REPRESENTATIONS

First of all, via the  $n$ -dimensional irreducible representation  $r_n : \mathrm{PGL}(2, \mathbf{C}) \hookrightarrow \mathrm{PGL}(n, \mathbf{C})$ , we always consider  $\chi(M, \mathrm{PGL}(2, \mathbf{C}))$  as a subset of  $\chi(M, \mathrm{PGL}(n, \mathbf{C}))$ . From now on, we denote this last space by  $\chi$ , as  $n \geq 3$  remains fixed.

We will always assume that our manifold  $M$  has only one cusp and therefore drop the index  $i$ : the peripheral torus is denoted  $T$  and  $l$  and  $m$  are its chosen longitude and meridian. This simplifies notations without hiding any difficulty. We will occasionally explain what should be adapted for the case of  $\nu$  cusps.

**2.1. Local rigidity in  $\chi$ .** Let  $\rho$  be the representative of an element  $[\rho]$  in  $\chi(M, \mathrm{PGL}(n, \mathbf{C}))$ . Menal-Ferrer and Porti [MFP11] proved the local rigidity around  $[\rho_{\mathrm{geom}}]$  in  $\chi(\pi_1(M), \mathrm{SL}(n, \mathbf{C}))$ :

**Theorem 1** (Menal-Ferrer and Porti). *Around  $[\rho_{\mathrm{geom}}]$ , the variety  $\chi(\pi_1(M), \mathrm{SL}(n, \mathbf{C}))$  is a complex manifold of dimension  $(n - 1)$  for*

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<sup>1</sup>I do not know a direct proof of it – that is without using Mostow rigidity. I think it would be very interesting to investigate if we might avoid it.

which the symmetric functions of the eigenvalues of the  $\rho(l)$  are local parameters.

In particular any  $[\rho]$  close enough of  $[\rho_{\text{geom}}]$  is completely determined by its peripheral representations.

**Remark 1.** *This result is proven when there are  $\nu$  cusps. In this case, the dimension of the manifold becomes  $(n - 1)\nu$ . For parameters, you have to choose a longitude in each peripheral torus and consider the symmetric functions of their eigenvalues.*

In this paper, we prefer to work with the group  $\text{PGL}(n, \mathbf{C})$  and the space denoted by  $\chi$ . Let us remark that the previous theorem translate directly into a theorem about  $\chi$  in the neighbourhood of the geometric representation by the following trick: choose a finite generating set  $S$  for  $\pi_1(M)$ . The geometric representation lifts to  $\text{SL}(n, \mathbf{C})$ . Hence for any representation  $\rho \in \text{Hom}(\pi_1(M), \text{PGL}(n, \mathbf{C}))$  close enough to  $\rho_{\text{geom}}$ , there is a unique lift  $\rho'(g) \in \text{SL}(n, \mathbf{C})$  of every  $\rho(g)$ ,  $g \in S$ , such that  $\rho'(g)$  is close to  $\rho_{\text{geom}}(g)$ . This defines the representation  $\rho' \in \text{Hom}(\pi_1(M), \text{SL}(n, \mathbf{C}))$  which is a lift of  $\rho$  and verifies that the eigenvalues of  $\rho'(l)$  and  $\rho'(m)$  are close to 1.

Hereafter, we abuse the notations and still speak about the symmetric functions of the eigenvalues of  $\rho$  whereas we should more precisely speak about those of  $\rho'$ .

**2.2. A ramified covering of  $\chi$ .** Let  $T$  be the boundary torus and  $l, m$  be the fixed longitude and meridian. It will be convenient for our purpose to fix the upper-triangular form of  $\rho(l)$  and  $\rho(m)$ , which amounts to choosing an order on their eigenvalues. This will be done classically by passing to a finite ramified covering of  $\chi$  describing the space of representations decorated by flags fixed by the peripheral representations.

Note that  $\rho(l)$  and  $\rho(m)$  commute. So they are simultaneously trigonalizable over  $\mathbf{C}$ . At  $\rho = \rho_{\text{geom}}$ , these matrices are unipotent and they have a regular upper-triangular form: there is a unique complete flag  $F_T$  in  $\mathbf{C}^n$  they fix. Stated in a more concrete way, they are conjugated to upper triangular matrices, with diagonal entries equal to 1 and non-zero entries above the diagonal (it is a unique Jordan bloc of size  $n$ ). After a small deformation, these matrices generically become simultaneously diagonalizable with distinct eigenvalues. Hence they fix  $n!$  different flags in  $\mathbf{C}^n$ . Each of these flags is near  $F_T$ , or in other terms you have a choice of  $n!$  distinct upper-triangular representatives for  $(\rho(l), \rho(m))$ .

Let  $\mathcal{F}$  be the space of complete flags in  $\mathbf{C}^n$ . Define now the space  $\chi'$  by:

$$\{(\rho, F) \in \text{Hom}(\pi_1(M), \text{PGL}(n, \mathbf{C})) \times \mathcal{F} \text{ such that } \rho(\pi_1(T)).F = F\} / \text{PGL}(n, \mathbf{C}).$$

There is a natural projection  $\chi' \rightarrow \chi$  given by forgetting about the flag. Moreover, as at  $[\rho_{\text{geom}}]$  the peripheral representation fixes only one flag (it is regular unipotent)  $F_T$ , the fiber of  $[\rho_{\text{geom}}]$  is exactly the point  $[\rho_{\text{geom}}, F_T]$ . For the sake of simplicity we will sometimes abuse notation and still denote  $[\rho_{\text{geom}}]$  this point of  $\chi'$ .

In some sense,  $\chi'$  is exactly the space where one may speak of the eigenvalues of  $\rho(l)$  and not only their symmetric functions: indeed, consider  $[\rho, F] \in \chi'$ . Then there are complex numbers  $L_k$  and  $M_k$  for  $1 \leq k \leq n-1$ , such that for any basis adapted to the flag  $F$ , the matrices  $\rho(l)$  and  $\rho(m)$  are simultaneously upper-triangular with:

$$(1) \quad \rho(l) = \begin{pmatrix} 1 & * & * & * & * \\ & L_1 & * & * & * \\ & & L_1 L_2 & * & * \\ & & & \ddots & * \\ & & & & L_1 \dots L_{n-1} \end{pmatrix}$$

and<sup>2</sup>

$$\rho(m) = \begin{pmatrix} 1 & * & * & * & * \\ & M_1 & * & * & * \\ & & M_1 M_2 & * & * \\ & & & \ddots & * \\ & & & & M_1 \dots M_{n-1} \end{pmatrix}.$$

This defines an application  $\text{Hol}_{\text{periph}}$  on the space  $\chi'^3$ :

$$\begin{aligned} \text{Hol}_{\text{periph}} &: \chi' \rightarrow (\mathbf{C}^{(n-1)})^2 \\ [\rho, F] &\mapsto ((L_k[\rho, F])_k, (M_k[\rho, F])_k) \end{aligned}$$

We claim that  $\chi'$  is a ramified covering of the space  $\chi$ , with covering group the Weyl group of  $\text{PGL}(n, \mathbf{C})$ . The latter is also the permutation group of  $n-1$  points. This ramified covering is often called the space of decorated representations ([BFG12a], for example). This claim is proven in the following fact and is a consequence of Menal-Ferrer and Porti theorem:

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<sup>2</sup>The  $L_k$ 's are given by the usual choice of simple positive roots for the diagonal torus associated to the upper triangular group.

<sup>3</sup>When there are different peripheral tori  $(T_i)_{i=1..n}$ , we define in the obvious way the functions  $L_k^i$ ,  $M_k^i$  and  $\text{Hol}_{\text{periph}}$ .

**Fact 1.** *The space  $\chi'$  is a ramified covering of  $\chi$  around  $[\rho_{\text{geom}}]$ .*

*Moreover the map  $\text{Hol}_{\text{periph}}$  is a local holomorphic embedding into  $\mathbf{C}^{2(n-1)}$ . Its image is locally a complex manifold of dimension  $(n-1)$ . The collection  $(L_k[\rho])_k$  is a local parameter for this submanifold in the neighbourhood of  $\text{Hol}_{\text{periph}}[\rho_{\text{geom}}]$ .*

**Remark 2.** *As for theorem 1, the generalisation to  $\nu$  cusps is valid, with the image of  $\text{Hol}_{\text{periph}}$  being locally a manifold of dimension  $(n-1)\nu$  in  $\mathbf{C}^{2(n-1)\nu}$*

As in the  $\text{PGL}(2, \mathbf{C})$ -case [Cho04], we really need  $M$  to be hyperbolic and to work in a neighbourhood of the geometric representation for this statement to hold. Note that we give in [BFG12a] an actual description (for  $n = 3$ ) of a neighbourhood of  $[\rho_{\text{geom}}]$  for which it holds (and counterexamples without the assumptions).

*Proof.* For  $1 \leq k \leq n-1$ , denote by  $\sigma_k : \mathbf{C}^n \rightarrow \mathbf{C}$  the  $k$ -th symmetric function of  $n$  complex numbers and, with a slight abuse of notations, still denote by  $\sigma_k$  the map which sends a matrix to the  $k$ -th symmetric function of its eigenvalues. Then we have two lines of applications:

$$\begin{array}{ccccc} \chi' & \xrightarrow{\text{Hol}_{\text{periph}}} & (\mathbf{C}^{n-1})^2 & \rightarrow & \mathbf{C}^{n-1} \\ [\rho, F] & \mapsto & ((L_k[\rho, F])_k, (M_k[\rho, F])_k) & \mapsto & (L_k[\rho, F])_k, \end{array}$$

and

$$\begin{array}{ccccc} \chi & \rightarrow & (\mathbf{C}^{n-1})^2 & \rightarrow & \mathbf{C}^{n-1} \\ [\rho] & \mapsto & ((\sigma_k(\rho(l)))_k, (\sigma_k(\rho(m)))_k) & \mapsto & (\sigma_k(\rho(l)))_k. \end{array}$$

In a neighbourhood of  $1 \in \mathbf{C}$ , we denote by  $z^{\frac{1}{n}}$  the branch of  $n$ -th root sending 1 to 1. We may then define, in a neighbourhood of  $(1, \dots, 1)$  in  $\mathbf{C}^{n-1}$  the map:

$$e : \begin{cases} \mathbf{C}^{n-1} & \rightarrow \\ (a_1, \dots, a_{n-1}) & \mapsto \frac{1}{(a_1^{n-1} a_2^{n-2} \dots a_{n-1})^{\frac{1}{n}}} (1, a_1, a_1 a_2, \dots, a_1 a_2 \dots a_{n-1}) \end{cases}$$

This map seems complicated mostly because of the choices in the definition of the  $L_k$ 's and  $M_k$ 's. We stick with this notation as they will be adapted for the remainder of the paper. In simple terms, this map describes the eigenvalues of  $\frac{1}{\det(A)^{\frac{1}{n}}} A$ , where  $A$  is the matrix:

$$A = \begin{pmatrix} 1 & * & * & * & * \\ & a_1 & * & * & * \\ & & a_1 a_2 & * & * \\ & & & \ddots & * \\ & & & & a_1 \dots a_{n-1} \end{pmatrix}.$$

By construction the two above lines fit into the commutative diagram:

$$\begin{array}{ccccc} \chi' & \xrightarrow{\text{Hol}_{\text{periph}}} & (\mathbf{C}^{n-1})^2 & \rightarrow & \mathbf{C}^{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \chi & \rightarrow & (\mathbf{C}^{n-1})^2 & \rightarrow & \mathbf{C}^{n-1} \end{array},$$

where the last two vertical arrows are constructed with the map

$$\begin{array}{ccc} \mathbf{C}^{n-1} & \rightarrow & \mathbf{C}^{n-1} \\ (a_1, \dots, a_{n-1}) & \mapsto & (\sigma_k(e(a_1, \dots, a_{n-1})))_{1 \leq k \leq n-1}. \end{array}$$

Now, what should we prove? The theorem 1 of Menal-Ferrer and Porti tells us the structure of the second line around  $[\rho_{\text{geom}}]$ : the first arrow biholomorphically sends an open set of  $\chi$  onto a submanifold of  $(\mathbf{C}^{n-1})^2$  of dimension  $\mathbf{C}^{n-1}$  and for which the second arrow gives local parameters. The last vertical arrow is the classical ramified covering with covering group the permutation group of  $n - 1$  points. So we just need to prove that the composition of the two arrows of the first line is injective: indeed, if this holds, the projection  $\chi' \rightarrow \chi$  is isomorphic to the classical ramified covering.

So consider  $[\rho] \in \chi$ , and  $(\sigma_k(\rho(l)))_k$  the vector of symmetric functions of the eigenvalues of  $\rho(l)$ . Let  $(L_1, \dots, L_{n-1})$  be a preimage of this vector by the third vertical arrow. Then we want to prove that there exists a *unique* flag  $F$ , such that  $[\rho, F]$  is sent to  $(L_1, \dots, L_{n-1})$ . In terms of the vector  $(L_1, \dots, L_{n-1})$ , the eigenvalues of  $\rho(l)$  are  $1, L_1, \dots, L_1 \cdots L_{n-1}$  (see eq. 1). If those are distinct, there is no problem:  $F$  has to be the flag whose  $k$ -dimensional space is the sum of the eigenlines associated to the  $k$  first eigenvalues  $1, L_1, \dots, L_1 \cdots L_k$ . This is the generic case.

But problems may occur when some eigenvalues coincide: for example if the eigenspace associated to 1 has dimension  $\geq 2$ , we would have different possible choices for the first line of  $F$ . We claim this does not happen in a neighbourhood of  $[\rho_{\text{geom}}]$ : even if an eigenvalue has multiplicity<sup>4</sup> greater than 2, its eigenspace will still be a line:

**Lemma 1.** *For  $\rho$  close enough to  $\rho_{\text{geom}}$ , for an eigenvalue  $\lambda$  of multiplicity  $r$  of  $\rho(l)$ , and an integer  $1 \leq k \leq r$ , there is a unique  $k$ -plane invariant by  $\rho(l)$  inside the characteristic space associated to  $\lambda$ .*

*Proof.* This amounts to say that the Jordan decomposition of  $\rho(l)$  is given by a unique block for each eigenvalue. This is equivalent to the

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<sup>4</sup>Let us precise the notations: the multiplicity of an eigenvalue is intended as its multiplicity as a root of the characteristic polynomial. We will say that an eigenvalue is simple if the dimension of its eigenspace is 1. Beware that an eigenvalue can at the same time be simple and have a multiplicity  $\geq 2$ .

fact that every eigenvalue is simple, i.e. its eigenspace is a line. This is true at  $[\rho_{\text{geom}}]$ , as  $\rho_{\text{geom}}(l)$  is regular unipotent, and it is an open condition given by the property: "the characteristic polynomial and its derivative are coprime polynomials". So it holds in a neighbourhood of  $[\rho_{\text{geom}}]$ .  $\square$

With this lemma, one concludes the proof: the  $k$ -dimensional space of  $F$  has to be constructed in the following way: consider the  $k$  first eigenvalues  $1, L_1, \dots, L_1 \cdots L_k$ . For each eigenvalue  $\lambda$  appearing, denote by  $r_\lambda$  the number of times it appears. And define the  $k$ -plane as the sum of the unique  $r_\lambda$ -planes invariant by  $\rho(l)$  inside the characteristic space associated to  $\lambda$ .  $\square$

**2.3. The lift of hyperbolic representations.** We keep track here of the image  $r_n(\chi(\pi_1(M), \text{PGL}(2, \mathbf{C})))$  in  $\chi$ . As said in the introduction of this section, we will simply denote by  $\chi(\pi_1(M), \text{PGL}(2, \mathbf{C}))$  this image.

**Fact 2.** *A point  $[\rho] \in \chi$  close enough to  $[\rho_{\text{geom}}]$  belongs to  $\chi(M, \text{PGL}(2, \mathbf{C}))$  if and only if there is a lift in the ramified covering  $\chi'$  such that, for all  $i$ , we have  $L_1[\rho] = L_2[\rho] = \dots = L_{n-1}[\rho]$ . In this case, we also have  $M_1(\rho) = M_2[\rho] = \dots = M_{n-1}[\rho]$ .*

*Proof.* An easy computation of

$$r_n \begin{pmatrix} t & * \\ 0 & t^{-1} \end{pmatrix}$$

shows that it is an upper triangular matrix with diagonal entries

$$t^{n-1}, t^{n-3}, \dots, t^{-(n-3)}, t^{-(n-1)}.$$

That is, with the notation of eq. 1,  $L_1 = \dots = L_{n-1} = t^{-2}$ . As locally the eigenvalues of  $\rho(l)$  determine a point in  $\chi'$ , the fact holds.  $\square$

**2.4. Peripheral discreteness.** This paper aims to understand the so-called *peripheral discreteness*: are the peripheral representations discrete or not? Looking for any global result is hopeless: as mentioned in the introduction, the  $\text{PGL}(2, \mathbf{C})$ -case is already understood and shows both non-discreteness (the generic feature in this dimension) and discreteness (for Dehn surgeries). So we try to understand the generic behaviour. Precisely, we prove in the  $\text{PGL}(n, \mathbf{C})$ -case that peripheral discreteness becomes the generic behaviour. Let  $\mathcal{U}$  be a neighbourhood of  $[\rho_{\text{geom}}]$  in  $\chi'$  on which  $\text{Hol}_{\text{periph}}$  is injective and the projection  $\chi' \rightarrow \chi$  is a ramified covering. Then we have:

**Theorem 2.** *Let  $M$  be a complete hyperbolic manifold of dimension 3 with 1 cusp. There is a real-analytic subvariety  $\mathcal{D}$  of  $\mathbf{C}^{(n-1)}$  of codimension  $\geq 1$  verifying that for any  $[\rho, F]$  in  $\mathcal{U}$ , with  $\text{Hol}_{\text{periph}}[\rho, F] =$*



$(L_k, M_k)_k$ , we have:  
 if  $(L_k)_k$  lies outside of  $\mathcal{D}$ , then the peripheral representation of  $\rho$  is discrete and faithful.

**Remark 3.** *The generalisation to  $\nu$  cusps is natural:  $\mathcal{D}$  becomes a subvariety of  $\mathbf{C}^{(n-1)\nu}$  of codimension  $\geq 1$ .*

It may seem surprising at first glance. But there is an heuristic evidence for this result. Indeed, the peripheral representations are representations of  $\mathbf{Z}^2$ . When in the  $\mathrm{PGL}(2, \mathbf{C})$ -case, outside of the geometric representation, both the elements  $\rho(l)$  and  $\rho(m)$  are loxodromic and preserve the same geodesic. Hence, this  $\mathbf{Z}^2$  naturally embeds in the stabilizer of this geodesic. The latter is a diagonal subgroup isomorphic to  $\mathbf{C}^*$ . At the end, you get some  $\mathbf{Z}^2$  included in  $\mathbf{C}^*$ . It is seldom discrete. Now, when the ambient group becomes  $\mathrm{PGL}(n, \mathbf{C})$  with  $n \geq 3$ , then the group  $\mathbf{Z}^2$  is (generically) mapped inside a diagonal subgroup which is isomorphic to  $(\mathbf{C}^*)^{n-1}$ . The higher rank indicates that the generic behaviour should be the discreteness and faithfulness<sup>5</sup>.

Our proof of the theorem will follow this heuristics. We will prove that, under the hypothesis of the theorem, no hidden algebraic relationship between  $\rho(l)$  and  $\rho(m)$  prevent the discreteness or the faithfulness. This proof is completed in section 3.2. Beware however that it is not some general triviality. Indeed, if we do not work in a neighbourhood of the geometric representation, one may exhibit examples of a strong and simple relationship. A counterexample is given by the 8-knot complement: when looking at the neighbourhood of another representation  $\rho$  whose peripheral representations are unipotent – actually lying inside  $\mathrm{PU}(2, 1)$  – such a simple relation holds and prevents faithfulness, see section 4.

We conclude this subsection by the criterion for discreteness and faithfulness we will use. It is an elementary fact, when you use the homeomorphism  $\mathbf{C}^* \simeq \mathbf{R}^* \times \mathbf{S}^1$ :

**Fact 3.** *Let  $[\rho, F]$  belongs to  $\mathcal{U}$  and note  $\mathrm{Hol}_{\mathrm{periph}}[\rho, F] = (L_k, M_k)_k$ . Then a sufficient condition for the restriction of  $\rho$  to  $\pi_1(T)$  to be discrete and faithful is:*

*There exist  $1 \leq k < h \leq n - 1$  such that*

$$\Delta_{k,h} := \det \begin{pmatrix} \log |L_k| & \log |L_h| \\ \log |M_k| & \log |M_h| \end{pmatrix} \neq 0.$$

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<sup>5</sup>Recall that when  $n = 2$ , outside of  $[\rho_{\mathrm{geom}}]$  but in a neighbourhood, the peripheral representation is *never* discrete and faithful.

*Proof.* For any representation  $[\rho, F]$  in  $\mathcal{U}$ , with  $\text{Hol}_{\text{periph}}[\rho, F] = (L_k, M_k)_k$ , the following are equivalent conditions all implying the discreteness and faithfulness of the restriction of  $\rho$  to  $\pi_1(T)$ :

- The vectors

$$(L_1, L_1 L_2, \dots, L_1 \cdots L_{n-1})$$

and

$$(M_1, M_1 M_2, \dots, M_1 \cdots M_{n-1})$$

generate a discrete subgroup of  $(\mathbf{C}^*)^{n-1}$  isomorphic to  $\mathbf{Z}^2$ .

- The vectors

$$(|L_1|, |L_1 L_2|, \dots, |L_1 \cdots L_{n-1}|)$$

and

$$(|M_1|, |M_1 M_2|, \dots, |M_1 \cdots M_{n-1}|)$$

generate a subgroup of  $(\mathbf{R}^*)^{n-1}$  isomorphic to  $\mathbf{Z}^2$ .

- The vectors

$$(\log |L_1|, \log |L_1 L_2|, \dots, \log |L_1 \cdots L_{n-1}|)$$

and

$$(\log |M_1|, \log |M_1 M_2|, \dots, \log |M_1 \cdots M_{n-1}|)$$

are free in  $\mathbf{R}^{n-1}$ .

The last point is translated in terms of non vanishing of at least one minor, that is one of the functions  $\Delta_{k,h}$ .  $\square$

Theorem 2 follows easily from this fact, at least if we may find one single representation  $\rho$  for which this determinant does not vanish. The subvariety  $\mathcal{D}$  is then defined by the vanishing of these determinants. A nice feature of the proof is that we will exploit what we know about the  $\text{PGL}(2, \mathbf{C})$ -case – even if in this case the determinants *always* vanish !

**2.5. Some facts about the  $\text{PGL}(2, \mathbf{C})$ -case.** The main result about the  $\text{PGL}(2, \mathbf{C})$ -case goes back to Thurston [Thu79] and is explained thoroughly in Neumann-Zagier's paper [NZ85]. We state it using our notations. Recall from facts 1 and 2 that the  $\text{PGL}(2, \mathbf{C})$ -case – seen as a subset of the ramified covering  $\chi'$  – is characterised by  $L_1 = \dots = L_{n-1}$ , which implies in turn  $M_1 = \dots = M_{n-1}$  for all  $i$ . Moreover, given a complex number  $L$  close enough to 1, there is a unique representation class  $[\rho_L] \in \chi(M, \text{PGL}(2, \mathbf{C})) \subset \chi(M, \text{PGL}(n, \mathbf{C}))$  such that  $L_1[\rho_L] = \dots = L_{n-1}[\rho_L] = L$ . For this representation, we denote by  $M(L)$  the common value of the  $M_k[\rho_L]$ . This defines a holomorphic map

$$L \mapsto M(L).$$

If  $L$  is close enough to 1, then  $[\rho_v]$  is close to  $[\rho_{\text{geom}}]$  and  $M(L)$  is close to 1. We choose the branch of  $\log$  which sends 1 to 0. The following theorem states that  $\log(M(L))$  is almost an affine function of  $\log(L)$  in the neighbourhood of 1 and its slope is the modulus of the euclidean structure of the torus  $T$  in the hyperbolic structure:

**Theorem 3** (Thurston). *There exists an analytic map  $\tau$  defined on a neighbourhood of 1 in  $\mathbf{C}$  such that we have :*

$$\log(M(L)) = \tau(L) \log(L).$$

*Moreover,  $\tau(1)$  belongs to the upper half-plane  $\mathbf{H}$  and is the modulus  $\mu$  of the euclidean structure on  $T$  given by the hyperbolic structure on the manifold  $M$  (associated to  $l, m$ ).*

Note that the modulus  $\mu$  is *not* a real number. This will prove useful at the end of the proof.

In light of fact 3, we are rather interested in the dependency of  $\log |M|$  in terms of  $\log |L|$ . However, for a countable set of complex numbers  $L$ , there exist two relatively prime integers  $p$  and  $q$  such that  $|L|^p |M|^q = 1$ . Indeed, for any neighbourhood of 1, for all but a finite number of relatively prime integers  $p, q$ , one may find an  $L$  such that this relation holds (see [NZ85, p. 322]). The dependency of these real parts is really a wild one and it is implied by the tameness of the complex dependency.

We will take advantage of that: we will show that the tamed behaviour of  $\log(M)$  with respect to  $\log(L)$  generalises to the  $\text{PGL}(n, \mathbf{C})$ -case. And this implies as a counterpart that the real parts dependencies are wild. The non-vanishing of the determinant of fact 3 will follow as a corollary.

### 3. COMPLEX ANALYTICITY AND ITS REAL COUNTERPART

**3.1. Ratios of complex logarithms.** Let  $\mathcal{V}$  be the projection on the space  $(L_k)_k$  of  $\text{Hol}_{\text{periph}}(\mathcal{U})$ : those are the possible vectors of eigenvalues of the longitude for a deformation of  $\rho_{\text{geom}}$ . From fact 1, we know that  $\mathcal{V}$  is a neighbourhood of  $(1, \dots, 1)$  and that to any  $v \in \mathcal{V}$ , there exists a unique point  $[\rho_v, F]$  of  $\mathcal{U}$  projecting to  $v$ .

We define, as in the  $\text{PGL}(2, \mathbf{C})$ -case, a map from  $\mathcal{V}$  to  $\mathbf{C}^{(n-1)}$  by

$$v = (L_k)_k \mapsto (M_k(v) := M_k[\rho_v, F])_k.$$

We first generalise the map  $\tau$  of the  $\text{PGL}(2, \mathbf{C})$ -case. The following proposition, which reduces to simple linear algebra, is the key point for theorem 2:

**Proposition 1.** *There exist  $(n - 1)$  applications  $\tau_k$ , holomorphic in a neighbourhood of  $(1, \dots, 1)$  in  $\mathcal{V}$ , such that we have, for all  $v = (L_k)_k \in \mathcal{V}$ :*

$$\log(M_k(v)) = \tau_k(v) \log(L_k).$$

*Moreover, for any  $k$ ,  $\tau_k(1, \dots, 1)$  is the modulus  $\mu$  of the euclidean structure on  $T$  induced by the hyperbolic structure on the manifold  $M$ .*

Let us note that the first part will only use the fact that we have local rigidity around  $[\rho_{\text{geom}}]$  and that  $\rho_{\text{geom}}(l)$  is regular unipotent (it is a unique Jordan bloc of size  $n$ ). Hence it generalises to more general settings: for example, when  $n = 3$ , I already mentioned that it is possible to make actual computations in order to find some representations of fundamental groups of hyperbolic 3-manifolds whose peripheral representations are unipotent. In [BFG12a] we gave a simple criterion of local rigidity and it is an easy task to check that the image of  $l$  is a unique Jordan block.

*Proof.* The proof of the existence of the  $\tau_k$  is similar to the  $\text{PGL}(2, \mathbf{C})$ -case [NZ85, Lemma 4.1]: we already know from fact 1 that the functions  $v \mapsto \log(M_k(v))$  are holomorphic and vanish at  $v = (1, \dots, 1)$ . In order to get the existence of the  $\tau_k$ 's, we have to show that for any  $k$ , the mere condition  $L_k = 1$  on the  $k$ -th entry of  $v$  implies that  $M_k = 1$  for the same entry. Indeed, this will prove that the ratio  $\log(M_k(v))/\log(L_k)$  is defined when  $L_k = 1$ . This ratio is the function  $\tau_k$ .

Choose some  $v \in \mathcal{V}$  such that, for some  $k$ ,  $L_k = 1$ ; let  $[\rho_v, F]$  be the representation class associated to  $v$ . We choose a basis  $(e_1, \dots, e_n)$  of  $\mathbf{C}^n$  adapted to the flag  $F$ . In this basis, the matrices  $\rho_v(l)$  and  $\rho_v(m)$  present the upper-triangular form we gave in eq. 1. We furthermore choose the basis so that  $\rho_v(l)$  is in Jordan form. For  $0 \leq k \leq n$ , let  $E_k$  be the  $k$ -dimensional subspace of  $\mathbf{C}^n$  generated by the first  $k$  vectors  $e_1, \dots, e_k$  (with  $E_0 = \{0\}$ ). Define  $A$ , resp  $B$ , to be the endomorphism of the plane  $E_{k+1}/E_{k-1}$  given by the action of  $\rho(l)$ , resp  $\rho(m)$ . Then we have, in the basis of  $E_{k+1}/E_{k-1}$  given by the projections of  $e_k$  and  $e_{k+1}$ , the following matrices representing  $A$  and  $B$  (recall that  $L_k = 1$ ):

$$A = L_1 \cdots L_{k-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } B = M_1 \cdots M_{k-1} \begin{pmatrix} 1 & y \\ 0 & M_k \end{pmatrix}.$$

Recall from lemma 1 that, for  $[\rho_v]$  close enough to  $[\rho_{\text{geom}}]$ , the eigenvalue  $L_1 \cdots L_{k-1}$  is simple. This implies that  $x \neq 0$ . But we also know that  $\rho_v(l)$  and  $\rho_v(m)$  commute. So do  $A$  and  $B$ . This implies  $M_k = 1$ .

The second point is given by the  $\text{PGL}(2, \mathbf{C})$ -case: as seen in sections 2.3 and 2.5, a point in the subvariety  $L_1 = \dots = L_{n-1}$  – common value hereafter denoted by  $L$  – corresponds to the representation  $\rho_L$  in

$\chi(M, \mathrm{PGL}(2, \mathbf{C}))$ . Hence we have  $M_1 = \dots = M_{n-1} = M$ . Hence, for any  $k$ , the ratio  $\tau_k(v) = \log(M_k)/\log(L_k)$  equals the function  $\tau(L) = \log(M)/\log(L)$  defined in the hyperbolic case. And for this  $\tau$ , theorem 3 gives  $\tau(1) = \mu$ .  $\square$

**3.2. Proof of the generic discreteness.** We are now ready to complete the proof of theorem 2. The idea is simple: you cannot have both a nearly linear relation with non real coefficients between the vectors  $(\log(L_k))_k$  and  $(\log(M_k))_k$  and a colinearity (over  $\mathbf{R}$ ) between  $(\log |L_k|)_k$  and  $(\log |M_k|)_k$ . As before let  $v$  be the vector  $(L_k)_k$  and define the notation  $\log(v) = \max_k |\log(L_k)|$ . We have:

$$\log(M_k) = \mu \log(L_k) + o(\log(v))$$

It yields the following approximation for

$$\Delta_{k,h} = \det \begin{pmatrix} \log |L_k| & \log |L_h| \\ \log |M_k| & \log |M_h| \end{pmatrix} :$$

$$\Delta_{k,h} = \mathrm{Im}(\mu) (\arg(L_k) \log |L_h| - \arg(L_h) \log |L_k|) + o(\log(v)^2).$$

We may look at a deformation defined, for  $t > 0$ , by  $L_k(t) = (1+t)e^{it}$  and  $L_h(t) = \overline{L_k}$  (all the other being fixed to 1). As  $\mathrm{Im}(\mu) \neq 0$ , we get the non vanishing of the determinant. This proves theorem 2.

**3.3.  $[\rho_{\mathrm{geom}}]$  is not a smooth point of the real analytic subvariety.** From the previous computation, it is clear that for any  $k, h$ , the differential at  $v = (1, \dots, 1)$  of

$$\Delta_{k,h} = \det \begin{pmatrix} \log |L_k| & \log |L_h| \\ \log |M_k| & \log |M_h| \end{pmatrix}$$

vanishes. Thus the local geometry of the subvariety  $\mathcal{D}$  of theorem 2, i.e. the vanishing locus of all determinants, is not clear. And it is a simple task to prove that  $(1, \dots, 1)$  is indeed a singular point on this subvariety, as the intersection of different branches: consider the deformation defined by all entries  $L_h$  fixed to 1, except one of them (say  $L_k$ ). Then all the  $M_h$  are constant equal to 1 except  $M_k$ . And all the determinants vanish.

So, in terms of tangent vectors, any vector with only one non-zero entry is tangent to the subvariety  $\mathcal{D}$  at  $(1, \dots, 1)$ . As  $\mathcal{D}$  is not of maximal dimension, it surely shows that  $(1, \dots, 1)$  is not a smooth point of this subvariety.

## 4. EXAMPLE : THE 8-KNOT COMPLEMENT

Building upon the work [BFG12b], a team works on a generalisation the famous computer program Snappea in order to understand representations in  $\mathrm{PGL}(3, \mathbf{C})$  of knot complements. Some results have already been mentioned in [BFG12a]. A forthcoming paper by Falbel, Koseleff and Rouillier will explain thoroughly how some parts of  $\chi(M, \mathrm{PGL}(3, \mathbf{C}))$  may be computed, at least for some  $M$ , the most worked-out example being the 8-knot complement.

Recall the well-known presentation of its fundamental group:

$$\langle g_1, g_3 | [g_3, g_1^{-1}]g_3 = g_1[g_3, g_1^{-1}] \rangle.$$

For this manifold, one get a complete list of representation whose peripheral holonomy is unipotent (see [Fal08] and more recently [DE13], which shows that  $\rho_2$  and  $\rho_3$  are intimately related). Up to some Galois conjugations there are only 4 of them:

- The holonomy  $[\rho_{\mathrm{geom}}]$  of the hyperbolic structure on  $M$ .
- $[\rho_1]$  defined on the generators by :

$$\rho_1(g_1) = \begin{pmatrix} 1 & 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_1(g_3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -1 & 1 \end{pmatrix}$$

- $[\rho_2]$  defined on the generators by :

$$\rho_2(g_1) = \begin{pmatrix} 1 & 1 & -\frac{1}{2} - \frac{i\sqrt{7}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_2(g_3) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{1}{2} + \frac{i\sqrt{7}}{2} & 1 & 1 \end{pmatrix}$$

- $[\rho_3]$  defined on the generators by :

$$\rho_3(g_1) = \begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_3(g_3) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{4} - \frac{i\sqrt{7}}{4} & 1 & 0 \\ -1 & -\frac{5}{4} - \frac{i\sqrt{7}}{4} & 1 \end{pmatrix}$$

All those representations may be checked to be locally rigid and their peripheral representations to be regular unipotent. Moreover, one may parametrize a neighbourhood of each of these representations in  $\chi(M, \mathrm{PGL}(3, \mathbf{C}))$  ; beware that the actual computation is not an easy task and will be thoroughly described elsewhere. It is nevertheless possible to estimate the determinant  $\Delta_{1,2}$  – the only one to compute, as  $n = 3$  – on a neighbourhood of these representations.

This shows the following behaviour:

- In the neighbourhood of  $[\rho_{\mathrm{geom}}]$  the result of this article holds!  
One may get a bit of additional information: the subvariety  $\mathcal{D}$

is locally diffeomorphic to the isotropic cone of a quadratic form on  $\mathbf{C}^2$ . It was already suggested by the approximation for  $\Delta_{1,2}$  used in the proof of the generic discreteness (section 3.2).

- In the neighbourhood of  $[\rho_1]$ , the discriminants are not identically 0, so the peripheral discreteness is still the generic case.
- In the neighbourhood of  $[\rho_2]$  and  $[\rho_3]$ , the determinants always vanish.

Note that, for  $\rho_2$  and  $\rho_3$  the peripheral representation is not faithful: for example for  $\rho_2$  the following relation holds between suitable chosen longitude  $l$  and meridian  $m$ :

$$\rho_2(l) = \rho_2(m)^5.$$

The computation shows that, in this case, not only the determinants vanish in a neighbourhood of  $[\rho_2]$ , but for each  $[\rho]$  close enough, we do still have:

$$\rho(l) = \rho(m)^5.$$

In other terms, the relation preventing the faithfulness of the peripheral representation at  $\rho_2$  is rigid.

It is tempting to think that, for a general manifold  $M$  and a discrete, locally rigid, representation  $\rho$  whose peripheral holonomy is regular unipotent, we should have:

the generic behaviour around  $[\rho]$  is the peripheral discreteness if and only if the peripheral representation is faithful. Moreover, in case it is not, the relation preventing the faithfulness is rigid.

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